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The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc[☆]

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Abstract

In this paper, we give the definition of iterated order to classify functions of fast growth in the unit disc, and investigate the growth of solutions of linear differential equations with analytic coefficients of iterated order in the unit disc. We obtain several results concerning the iterated order of solutions.

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1. Definitions and introduction

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value-distribution theory of meromorphic functions in the unit disc $\Delta = \{z: |z| < 1\}$ (see [3,6]). In addition, let us recall the following definitions.

Definition A. [4] The order of meromorphic function f in Δ is defined by

$$\sigma(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{\log T(r, f)}{\log \frac{1}{1-r}}; \quad (1.1)$$

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for an analytic function f in Δ , we also define

$$\sigma_M(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{\log \log M(r, f)}{\log \frac{1}{1-r}}, \quad (1.2)$$

where $M(r, f)$ is the maximum modulus function.

Remark 1.1. M. Tsuji [5, Theorem V.13] gives that if f is an analytic function in Δ , then

$$\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1. \quad (1.3)$$

Observe that there exists f such that $\sigma(f) \neq \sigma_M(f)$; for example, a function $g(z) = \exp\{(1-z)^{-a}\}$ ($a > 1$) satisfies $\sigma(g) = a - 1$ and $\sigma_M(g) = a$, see [5, p. 205]. From this, we see that $\sigma(f) = \sigma_M(f)$ if $\sigma(f) = \infty$. So it is natural to ask the following question:

Question 1. How to describe precisely the fast growth of infinite order of functions in the Δ ?

Definition B. [4] Let f be analytic in Δ , and let $q \in [0, \infty)$. Then f is said to belong to the weighted Hardy space H_q^∞ provided that

$$\sup_{z \in \Delta} (1 - |z|^2)^q |f(z)| < \infty.$$

We say that f is an H -function when $f \in H_q^\infty$ for some q .

Definition C. [2] Let f be an H -function and set

$$p = \inf\{q \geq 0: f \in H_q^\infty\}.$$

Then f is said to belong to the space G_p .

We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane \mathbb{C} . There are many types of definitions of small growth order of functions in Δ (i.e., see [2,7]).

Definition 1.1. Let f be a meromorphic function in Δ , and

$$D(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = b. \quad (1.4)$$

If $b < \infty$, we say that f is of finite b degree (or is nonadmissible); if $b = \infty$, we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r, f)$.

Definition 1.2. Let f be an analytic function in Δ ; if

$$D_M(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{\log M(r, f)}{\log \frac{1}{1-r}} = a < \infty \quad (\text{or } = \infty), \quad (1.5)$$

then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r, f)$.

Remark 1.2. It can be deduced that the constant p in Definition C satisfies $p = D_M(f) = a$, which is denoted as $f \in D_a$ in [7, Definition 5].

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in \mathbf{C} (see [1,11]). Thus we answer Question 1 from the definitions. Let us define inductively, for $r \in [0, 1)$, $\exp^{[1]} r = e^r$ and $\exp^{[n+1]} r = \exp(\exp^{[n]} r)$, $n \in \mathbf{N}$. For all r sufficiently large in $(0, 1)$, we define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r)$, $n \in \mathbf{N}$. We also denote $\exp^{[0]} r = r = \log^{[0]} r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$. Moreover, we denote by E and H subsets in $[0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$ and $\int_H \frac{dr}{1-r} = \infty$, respectively. They may be different in various instances.

Definition 1.3. The iterated n -order $\sigma_n(f)$ of a meromorphic function $f(z)$ in Δ is defined by

$$\sigma_n(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{\log^{[n]} T(r, f)}{\log \frac{1}{1-r}} \quad (n \in \mathbf{N}); \quad (1.6)$$

for an analytic function f in Δ , we also define

$$\sigma_{M,n}(f) = \overline{\lim}_{r \rightarrow 1^-} \frac{\log^{[n+1]} M(r, f)}{\log \frac{1}{1-r}} \quad (n \in \mathbf{N}), \quad (1.7)$$

Remark 1.3. (1) If $n = 1$, then we denote $\sigma_1(f) =: \sigma(f)$, $\sigma_{M,1}(f) =: \sigma_M(f)$.

(2) If $n = 2$, then denote by $\sigma_2(f)$ the hyperorder (see [8]).

Definition 1.4. The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is nonadmissible,} \\ \min\{n \in \mathbf{N}: \sigma_n(f) < \infty\} & \text{if } f \text{ is admissible,} \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbf{N}. \end{cases}$$

For an analytic function f in Δ , we also define

$$i_M(f) = \begin{cases} 0 & \text{if } f \text{ is nonadmissible,} \\ \min\{n \in \mathbf{N}: \sigma_{M,n}(f) < \infty\} & \text{if } f \text{ is admissible,} \\ \infty & \text{if } \sigma_{M,n}(f) = \infty \text{ for all } n \in \mathbf{N}. \end{cases}$$

Remark 1.4. If $\sigma_n(f) < \infty$ or $i(f) \leq n$, then we say that f is of finite n -order; if $\sigma_n(f) = \infty$ or $i(f) > n$, then we say that f is of infinite n -order. In particular, we say that f is of finite order if $\sigma(f) < \infty$ or $i(f) \leq 1$; f is of infinite order if $\sigma(f) = \infty$ or $i(f) > 1$.

Now we give two propositions about the above defined characteristics.

Proposition 1.1. If f and g are meromorphic functions in Δ , $n \in \mathbf{N}$, then we have

- (i) $\sigma_n(f) = \sigma_n(1/f)$, $\sigma_n(a \cdot f) = \sigma_n(f)$ ($a \in \mathbf{C} - \{0\}$);
- (ii) $\sigma_n(f) = \sigma_n(f')$;
- (iii) $\max\{\sigma_n(f+g), \sigma_n(f \cdot g)\} \leq \max\{\sigma_n(f), \sigma_n(g)\}$;
- (iv) if $\sigma_n(f) < \sigma_n(g)$, then $\sigma_n(f+g) = \sigma_n(g)$, $\sigma_n(f \cdot g) = \sigma_n(g)$.

Proof. (i) By Definition 1.3 and the First Main Theorem in Δ , it holds obviously.

(ii) Let $r_1 = r + \frac{1-r}{4}$, $r_2 = r + \frac{1-r}{2}$, $r_3 = r + \frac{3(1-r)}{4}$; similar discussion as to the inequality of C.T. Chuang (see [6, Theorem 4.1] or [12]), one can see that: if f is meromorphic in Δ , then for $r \rightarrow 1^-$, we have

$$T(r, f) < O\left(T(r_3, f') + \log \frac{1}{1-r}\right).$$

On the other hand,

$$\begin{aligned} T(r, f') &= m(r, f') + N(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right) + 2N(r, f) \\ &\leq 2T(r, f) + m\left(r, \frac{f'}{f}\right). \end{aligned}$$

Hence $\sigma_n(f) = \sigma_n(f')$.

Considering that

$$T(r, f+g) \leq T(r, f) + T(r, g), \quad T(r, f \cdot g) \leq T(r, f) + T(r, g) + O(1),$$

and

$$T(r, g) = T\left(r, \frac{f \cdot g}{f}\right), \quad T(r, g) = T(r, (f+g) - f),$$

we can obtain (iii) and (iv). \square

Proposition 1.2. *If f and g are analytic functions in Δ , $n \in \mathbf{N}$, then we have*

- (i) $\sigma_{M,n}(a \cdot f) = \sigma_{M,n}(f)$ ($a \in \mathbf{C} - \{0\}$);
- (ii) $\sigma_{M,n}(f) = \sigma_{M,n}(f')$;
- (iii) $\max\{\sigma_{M,n}(f+g), \sigma_{M,n}(f \cdot g)\} \leq \max\{\sigma_{M,n}(f), \sigma_{M,n}(g)\}$;
- (iv) if $\sigma_{M,n}(f) < \sigma_{M,n}(g)$, then $\sigma_{M,n}(f+g) = \sigma_{M,n}(g)$;
- (v) $D(f) \leq D_M(f)$;
- (vi) if $i(f) = n = 1$, then $\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1$; if $i(f) = n > 1$, then $\sigma_n(f) = \sigma_{M,n}(f)$;
- (vii) $i(f) = i_M(f)$.

Proof. By Definition 1.3, (i) holds obviously. From $|f+g| \leq |f| + |g| \leq 2 \max\{|f|, |g|\}$ and $|fg| \leq |f||g| \leq \max\{|f|, |g|\}^2$, we can get (iii) and (iv) considering that $g = (f+g) - f$. Now we prove (ii) and (v)–(vii).

(ii) For $|z| = r \in (0, 1)$, take point $z_0 = re^{i\theta}$ satisfying $|f'(z_0)| = M(r, f')$ and $s(r) = 1 - \frac{1}{2}(1-r)$, and a circle $C_r = \{\zeta: |\zeta - z_0| = s(r) - r\}$. Since

$$f'(z_0) = \frac{1}{2\pi} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \quad \text{and} \quad \max\{|f|: \zeta \in C_r\} \leq M(s(r), f),$$

we deduce that

$$M(r, f) = |f'(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)}{(\zeta - z_0)^2} (s(r) - r) d\theta \leq \frac{M(s(r), f)}{s(r) - r} = \frac{M(s(r), f)}{\frac{1}{2}(1-r)}.$$

Hence

$$\sigma_{M,n}(f') = \overline{\lim}_{r \rightarrow 1^-} \frac{\log^{[n+1]} M(r, f')}{\log \frac{1}{1-r}} \leq \overline{\lim}_{r \rightarrow 1^-} \frac{\log^{[n+1]} M(s(r), f)}{\log \left\{ \frac{1}{1-s(r)} \left(1 + \frac{\log \frac{1-s(r)}{1-r}}{\log \frac{1}{1-s(r)}} \right) \right\}} = \sigma_{M,n}(f).$$

On the other hand, from the formula

$$f(z) = \int_0^z f'(t) dt + f(0),$$

we obtain that

$$|f(z)| \leq \int_0^r |f'(t)| dt + |f(0)| \leq \int_0^r M(t, f') dt + |f(0)| \leq M(r, f') + |f(0)|$$

for $|z| = r$. Hence $M(r, f) \leq M(r, f') + |f(0)|$, thus $\sigma_{M,n}(f) = \sigma_{M,n}(f')$. Therefore (ii) holds.

(v) From Nevanlinna's theory, we know that if $f(z)$ is analytic at $|z| = r < 1$, then

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{1+r}{1-r} (T(r, f)). \quad (1.8)$$

So $D(f) \leq D_M(f)$.

(vi) From (1.8), we get that

$$\frac{\log^{[n]} T(r, f)}{\log \frac{1}{1-r}} \leq \frac{\log^{[n+1]} M(r, f)}{\log \frac{1}{1-r}} \leq \frac{\log^{[n]} \left\{ \frac{1+r}{1-r} T(r, f) \right\}}{\log \frac{1}{1-r}}.$$

Thus we can see that $\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1$ if $n = 1$, and that $\sigma_n(f) = \sigma_{M,n}(f)$ if $n > 1$.

(vii) From (vi) and Definition 1.4, it is obvious that $i(f) = i_M(f)$. \square

Remark 1.5. (i) We have $\sigma_p(f) = \sigma_{M,p}(f)$ when $i(f) = i_M(f) = p > 1$, which is the same as the result of the iterated order (see [1, 11]) of an entire function in the complex plane \mathbb{C} .

(ii) We have $D(f) \leq D_M(f)$ if f is an analytic function in Δ . However it is not true that “ $D_M(f) < \infty$ if and only if $D(f) < \infty$ ” (see [7, Proposition 1]). For example, for the analytic function $f(z) = e^{g(z)} = e^{1/(1-z)}$ in Δ , one can get that $D_M(f) = \infty$ and $D(f) < \infty$.

2. Results of differential equations

Considering the growth of order of solutions of linear differential equations

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_0(z)f = 0, \quad (*)$$

where the coefficients $a_j(z)$ ($j = 0, \dots, k-1$) are analytic functions in Δ , there exist few results (see [4, 8, 9]) of precise estimation of the order of solutions of (*) because the Wiman–Valiron theory, which plays a very important role in the proof of estimations of order of solutions of equations on the whole complex plane, does not hold in Δ .

Theorem A. [4] Let $a_0(z), \dots, a_{k-1}(z)$ be the sequence of coefficients of (*) analytic in Δ . Let $a_j(z)$ be the last coefficient not being an H -function while the coefficients $a_{j+1}(z), \dots, a_{k-1}(z)$ are H -functions. Then (*) possesses at most j linearly independent analytic solutions of finite order of growth in Δ .

Theorem B. [9] Let $a_0(z), \dots, a_{k-1}(z)$ be the coefficients of (*) analytic in Δ satisfying that either $\sigma(a_j) < \sigma(a_0)$ ($j = 1, \dots, k-1$) or a_0 is admissible, and a_j ($j = 1, \dots, k-1$) are non-admissible. Then every solution $f \not\equiv 0$ is of infinite order.

Theorem C. [8] Consider equation

$$f'' + a_1(z)f' + a_0(z)f = 0, \quad (**)$$

where the coefficients $a_i(z)$ ($i = 0, 1$) with $\sigma_M(a_1) < \sigma_M(a_0)$ are analytic functions in Δ . Then every non-trivial solution f satisfies

$$\overline{\lim}_{r \rightarrow 1^-} \frac{\log \log T(r, f)}{\log \frac{1}{1-r}} \geq \sigma_M(a_0).$$

Thus it is a natural question as follows:

Question 2. Can we get the result

$$\overline{\lim}_{r \rightarrow 1^-} \frac{\log \log T(r, f)}{\log \frac{1}{1-r}} \leq \sigma_M(a_0)$$

in Theorem C?

In 2004, Z.-X. Chen and K.H. Shon [7] obtained some results of the small growth of solutions of (*) and (**) in Δ when $i(a_j) = 0$ ($j = 0, 1, \dots, k-1$). Thus there exists a natural question as follows:

Question 3. How about the iterated order of the fast growth of solutions of (*) or (**) in Δ ?

Let us consider homogeneous linear differential equations of the form

$$L(f) = f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0 \quad (k \in \mathbf{N}), \quad (2.1)$$

where the coefficients $A_j(z)$ ($j = 0, \dots, k-1$) are analytic functions in Δ , and at least one of them not constant. We exclude the case $A_j = \text{constant}$ ($j = 0, \dots, k-1$) because it is very well known. One knows that each solution of Eq. (2.1) is analytic in Δ . In what follows, we use the following notations.

Notation 2.1.

$$\begin{aligned} \delta &= \sup\{i(f): L(f) = 0\}, \\ \gamma_n &= \sup\{\sigma_n(f): L(f) = 0\}, \\ \gamma_{M,n} &= \sup\{\sigma_{M,n}(f): L(f) = 0\}, \\ p &= \max\{i(A_j): j = 0, \dots, k-1\}. \end{aligned}$$

Notation 2.2. If $0 < p < \infty$, then mark

$$\begin{aligned} \alpha &= \max\{\sigma_p(A_j): j = 0, \dots, k-1\}, \\ \alpha_M &= \max\{\sigma_{M,p}(A_j): j = 0, \dots, k-1\}. \end{aligned}$$

Remark 2.1. By Proposition 1.2(vii), we see that

$$\delta = \sup\{i_M(f): L(f) = 0\},$$

$$p = \max\{i_M(A_j): j = 0, \dots, k-1\}.$$

Remark 2.2. By Proposition 1.2(vi), we see that $\gamma_\delta = \gamma_{M,\delta}$ if $\delta > 1$, and that $\gamma_1 \leq \gamma_{M,1} \leq \gamma_1 + 1$ if $\delta = 1$.

Remark 2.3. By Proposition 1.2(vi), we see that $\alpha_M \geq \alpha$. In particular, $\alpha_M = \alpha$ if $p > 1$.

Now we give our main results as follows.

Theorem 2.1. For Eq. (2.1), the following conditions are satisfied:

- (i) $\delta \leq 1 + p$;
- (ii) if $0 < p < \infty$, then $\delta = 1 + p$ and $\alpha_M \geq \gamma_{M,p+1} = \gamma_{p+1} \geq \alpha$;
- (iii) if $p = 0$ and $\max\{D_M(A_j): j = 0, \dots, k-1\} = m$, then $\gamma_1 \leq \gamma_{M,1} \leq 1 + m$.

Corollary 2.1. For Eq. (2.1), if $1 < p < \infty$, then $\delta = 1 + p$ and $\alpha_M = \gamma_{M,p+1} = \gamma_{p+1} = \alpha$.

Theorem 2.2. If $0 < p < \infty$ and $j = \max\{n: i(A_n) = p, n = 1, \dots, k-1\}$, then Eq. (2.1) possesses at most j linearly independent solutions f with $i(f) \leq p$.

If the last coefficient $A_0(z)$ in Eq. (2.1) is the dominant coefficient, we know more about the iterated order of the growth of the solutions.

Theorem 2.3. Let $0 < p < \infty$ and $i(A_0) = p$. If $\max\{i(A_j): j = 1, \dots, k-1\} < p$ or $\max\{\sigma_{M,p}(A_j): j = 1, \dots, k-1\} < \sigma_{M,p}(A_0)$, then $i(f) = p+1$ and $\sigma_{M,p+1}(f) = \sigma_{p+1}(f) = \sigma_{M,p}(A_0) \geq \sigma_p(A_0)$ hold for all solutions $f \not\equiv 0$ of Eq. (2.1).

Theorem 2.4. Let $0 < p < \infty$ and $i(A_0) = p$. If $\max\{i(A_j): j = 1, \dots, k-1\} < p$ or $\max\{\sigma_p(A_j): j = 1, \dots, k-1\} < \sigma_p(A_0)$, then $i(f) = p+1$ and $\sigma_p(A_0) \leq \sigma_{M,p+1}(f) = \sigma_{p+1}(f) \leq \alpha_M$ hold for all solutions $f \not\equiv 0$ of Eq. (2.1).

Corollary 2.2. Let $1 < p < \infty$ and $i(A_0) = p$. If A_i ($i = 0, 1, \dots, k-1$) satisfy the conditions of Theorem 2.3 or Theorem 2.4, then $i(f) = p+1$ and $\sigma_{M,p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_0) = \sigma_{M,p}(A_0) = \alpha = \alpha_M$ hold for all solutions $f \not\equiv 0$ of Eq. (2.1).

Considering the second order equation

$$f'' + A_1(z)f' + A_0(z)f = 0, \tag{2.2}$$

we have the following results.

Theorem 2.5. For Eq. (2.2), $\delta = 1 + p$. In addition,

- (i) $\gamma_1 + 1 \geq \gamma_{M,1} \geq \gamma_1 \geq \frac{D(A_0)}{4} - 2$, if $p = 0$;
- (ii) $\gamma_1 + 1 \geq \gamma_{M,1} \geq \gamma_1 \geq D(A_1) - 2$, if $p = 0$.

Corollary 2.3. For Eq. (2.2), if $\infty > p > 1$, then $\delta = 1 + p$ and $\alpha_M = \gamma_{M,p+1} = \gamma_{p+1} = \alpha$.

3. Lemmas for the proofs of theorems

Lemma 3.1. [4] Let f be a meromorphic function in the unit disc and let $k \in \mathbf{N}$. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad (3.1)$$

where $S(r, f) = O(\log^+ T(r, f)) + O(\log(\frac{1}{1-r}))$, $r \notin E$, where $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 3.2. [1] Assume that $G = I \times J \subset \mathbf{C}$ is an open rectangle and $|A_j(z)| < b_j$ ($j = 0, \dots, k-1$) for all $z \in G$. Define

$$M = \frac{1}{2} [1 + \max\{2b_0 + b_1 + \dots + b_{k-1}, 1 + b_2, 1 + b_3, \dots, 1 + b_{k-2}\}].$$

Let z_0 be a point in G and f be a solution of equation

$$L(f) = f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0 \quad (k \in \mathbf{N}),$$

where the coefficients $A_j(z)$ ($j = 0, \dots, k-1$) are analytic functions on G . Let

$$\|f(z)\| = \{|f(z)|^2 + |f'(z)|^2 + \dots + |f^{(k-1)}(z)|^2\}^{1/2},$$

and $h(z) = \exp\{M \cdot (|\operatorname{Re} z - \operatorname{Re} z_0| + |\operatorname{Im} z - \operatorname{Im} z_0|)\}$. Then

$$\|f(z_0)\| \cdot (h(z))^{-1} \leq \|f(z)\| \leq \|f(z_0)\| \cdot h(z) \quad (3.2)$$

for every $z \in G$.

Lemma 3.3. Let E be a subset of $[0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$. If $F, G: (0, 1) \rightarrow \mathbf{R}$ are functions satisfying

- (i) F is nondecreasing and G is positive,
- (ii) $\lim_{r \rightarrow 1^-} G(s(r))/G(r) = 1$ for all functions $s: (0, 1) \rightarrow \mathbf{R}$ such that $0 < s(r) - r < \varepsilon$ ($r > 0$).

Then

$$\overline{\lim}_{r \rightarrow 1^-} \frac{F(r)}{G(r)} = \sup \left\{ \overline{\lim}_{n \rightarrow \infty} \frac{F(r_n)}{G(r_n)} : r_1 < \dots < r_n < \dots, r_n \rightarrow 1^- \text{ and } r_n \notin E \text{ } (n = 1, 2, \dots) \right\}. \quad (3.3)$$

Proof. We have

$$\overline{\lim}_{r \rightarrow 1^-} \frac{F(r)}{G(r)} = \sup \left\{ \overline{\lim}_{n \rightarrow \infty} \frac{F(r_n)}{G(r_n)} : \{r_n\}_{n=1}^\infty \in D \right\},$$

where $D = \{\{r_n\}_{n=1}^\infty: r_1 < r_2 < \dots, r_n \rightarrow 1^-\}$. Take $\{r_n\}_{n=1}^\infty \in D$ such that $\overline{\lim}_{n \rightarrow \infty} F(r_n)/G(r_n) = \overline{\lim}_{r \rightarrow 1^-} F(r)/G(r)$. One can find a sequence $\{r_n\}_{n=1}^\infty$ and an integer n_0 with $s_1 < s_2 < \dots < s_n < \dots$, $s_n \notin E$ for every n , and $0 < s_n - r_n < \varepsilon$ ($n > n_0$), because $\int_E \frac{dr}{1-r} < \infty$. Then $\{s_n\}_{n=1}^\infty \in D$ and $\lim_{n \rightarrow \infty} G(s_n)/G(r_n) = 1$, so

$$\overline{\lim}_{n \rightarrow \infty} \frac{F(r_n)}{G(r_n)} = \overline{\lim}_{n \rightarrow \infty} \frac{F(r_n)}{G(s_n)} \leq \overline{\lim}_{n \rightarrow \infty} \frac{F(s_n)}{G(s_n)} \leq \overline{\lim}_{r \rightarrow 1^-} \frac{F(r)}{G(r)},$$

and then (3.3) holds. \square

Lemma 3.4. Let f be an analytic function in Δ for which: either (i) $i_M(f) = b$ ($0 < b < \infty$) and $\sigma_{M,b}(f) = \sigma$, or (ii) $i_M(f) = b = 0$ and $D_M(f) = \sigma$. Then there exists a set $H \subset [0, 1)$ with $\int_H \frac{dr}{1-r} = \infty$, for $r \in H$, given $\varepsilon > 0$, we have

$$M(r, f) \geq \exp^{[b]} \left\{ \left(\frac{1}{1-r} \right)^{\sigma-\varepsilon} \right\}. \quad (3.4)$$

Proof. Set $\beta = \sigma - \varepsilon$, $\tau = \sigma - \varepsilon/2$; then there exists $\{r_n\} \subset [0, 1)$ satisfying $(1 - r_n)^{-\varepsilon/2} \geq n^\beta$ and $\log^{[b]} M(r, f) \geq (1 - r_n)^{-\tau}$. Hence for all $n \in \mathbb{N}$, we have

$$\log^{[b]} M(r, f) \geq \left(\frac{n}{1-r_n} \right)^\beta.$$

Set $H = \bigcup H_n$, $H_n = [r_n, 1 - \frac{1-r_n}{n}]$, if $r \in H$, then

$$\log^{[b]} M(r, f) \geq \log^{[b]} M(r_n, f) \geq \left(\frac{n}{1-r_n} \right)^\beta \geq \left(\frac{1}{1-r_n} \right)^\beta$$

and

$$\int_H \frac{dr}{1-r} \geq \int_{r_n}^{1-r_n/n} \frac{dr}{1-r} = \log n \rightarrow \infty \quad (n \rightarrow \infty).$$

Thus the lemma holds. \square

Lemma 3.5. [2] Let f be a meromorphic function in Δ . Let $\alpha \in (1, \infty)$ and $\beta \in (1, \infty)$ be constants, and k, j be integers satisfying $k > j \geq 0$. Assume that $f^{(j)} \not\equiv 0$. Let $\{a_m\}$ denote the sequence of all the zeros and poles of $f^{(j)}$ listed according to multiplicities and ordered by increasing moduli, and let $n_j(r)$ denote the counting function of the points $\{a_m\}$.

Then the following two statements holds:

(a) If $\{a_m\}$ is a finite sequence, then there exist constants $R \in (0, 1)$ and $C \in (0, \infty)$, such that for all z satisfying $R < |z| < 1$, we have (with $r = |z|$)

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq C \left[\frac{T(1 - \beta(1-r), f) - \log(1-r)}{(1-r)^2} \right]^{k-j}. \quad (3.5)$$

(b) If $\{a_m\}$ is an infinite sequence, then there exists an infinite sequence of discs $D_i = \{z: |z - c_i| < R_i\} \subset \Delta - \{0\}$ ($i = 1, 2, \dots$), such that

$$\sum_{i=1}^{\infty} \frac{R}{1 - |c_i|} < \infty,$$

and there exist constants $R \in (0, 1)$ and $C \in (0, \infty)$, such that for all z satisfying $z \notin \bigcup_{i=1}^{\infty}$ and $R < |z| < 1$, we have (with $r = |z|$)

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq C \left[\frac{T(1 - \beta(1 - r), f) - \log(1 - r)}{(1 - r)^2} + W(r) \right]^{k-j}, \quad (3.6)$$

where

$$W(r) = \frac{n_j(1 - \beta(1 - r))}{1 - r} \left(\log \frac{1}{1 - r} \right)^{\alpha} \log^+ n_j(1 - \beta(1 - r)).$$

- (c) There exists a set $E' \subset [0, 2\pi)$ which has linear measure zero, and a constant $C > 0$ such that if $\theta \in [0, 2\pi) - E'$, then there is a constant $R = R(\theta) \in [0, 1)$ such that for all z satisfying $\arg z = \theta$ and $R \leq |z| < 1$, we have that (3.5) or (3.6) holds, depending on whether $\{a_m\}$ is a finite or infinite sequence, respectively.

Lemma 3.6. [2] Let f be a meromorphic function in Δ of finite order σ . Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Assume that $f^{(j)} \not\equiv 0$. Then the following two statements holds:

- (a) There exists a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$ such that for all $z \in \Delta$ satisfying $|z| \notin E$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq \frac{1}{(1 - |z|)^{(k-j)(\sigma+2+\varepsilon)}}. \quad (3.7)$$

- (b) There exists a set $E' \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E'$, then there is a constant $R = R(\theta) \in [0, 1)$ such that for all z satisfying $\arg z = \theta$ and $R \leq |z| < 1$, we have that (3.7) holds.

4. Proof of Theorem 2.1

- (i) Let f be a solution of (2.1). Set

$$M(r) = \max\{|f(z)| : |z| = r\}, \quad M_j(r) = \max\{|A_j(z)| : |z| = r\};$$

$$N(r) = \sup\{|f(z)| : |z| \in G_r\} \quad \text{and} \quad N_j(r) = \sup\{|A_j(z)| : |z| \in G_r\},$$

where $G_r = (-r, r) \times (-r, r) \subset \Delta$, $j = 0, \dots, k-1$. It is obvious that one can replace $M(r)$ by $N(r)$, and $M_j(r)$ by $N_j(r)$ in the respective expressions of iterated n -order.

If $p = \infty$, then (i) is trivial. Otherwise, there exist real numbers $1 > r_0 > 0$, $d_0, \dots, d_{k-1} > 0$ such that $N_j(r) < \exp^{[p]} \left\{ \left(\frac{1}{1-r} \right)^{d_j} \right\}$ ($j = 0, \dots, k-1$), when $1 > r > r_0$. Apply Lemma 3.2 for z_0 and G_r to obtain

$$|f(z)| \leq \|f(z)\| \leq \|f(0)\| \cdot \exp\{2r + 2rN_0(r) + \dots + 2rN_{k-1}(r)\},$$

if $z \in G_r$ and $r > r_0$, because $M \leq 1 + N_0(r) + \dots + N_{k-1}(r)$. If $d \in (1 + \max\{d_0, \dots, d_{k-1}\}, \infty)$, one can find $1 > r_1 > r_0$ with

$$2r + 2rN_0(r) + \dots + 2rN_{k-1}(r) < \exp^{[p]} \left\{ \left(\frac{1}{1-r} \right)^d \right\},$$

when $1 > r > r_1$. If $s \in (d, \infty)$, then there is $1 > r_2 > r_1$ such that

$$\|f(0)\| \exp\{2r + 2rN_0(r) + \dots + 2rN_{k-1}(r)\} < \exp^{[p+1]} \left\{ \left(\frac{1}{1-r} \right)^s \right\},$$

when $1 > r > r_2$. Thus

$$N(r) < \exp^{[p+1]} \left\{ \left(\frac{1}{1-r} \right)^s \right\} \quad \text{and} \quad \frac{\log^{[p+2]} N(r)}{\log \frac{1}{1-r}} < s \quad \text{asymptotically.}$$

Hence $\sigma_{M,p+1}(f) < \infty$ and $i_M(f) \leq 1 + p$. So $\delta \leq 1 + p$.

(ii) Set $0 < p < \infty$. From the definition of α_M , it is clear that, for any $\varepsilon > 0$, one has $N_j(r) < \exp^{[p]} \{ (\frac{1}{1-r})^{\alpha_M + \varepsilon} \}$ for $r > r_0$. But it is easily derived that there is $1 > r_1 > r_0 \geq 0$ with

$$\|f(0)\| \exp\{2r + 2rN_0(r) + \cdots + 2rN_{k-1}(r)\} < \exp^{[p+1]} \left\{ \left(\frac{1}{1-r} \right)^{\alpha_M + 2\varepsilon} \right\}$$

when $1 > r > r_1$. Hence $N(r) < \exp^{[p+1]} \{ (\frac{1}{1-r})^{\alpha_M + 2\varepsilon} \}$ asymptotically. Hence

$$\frac{\log^{[p+2]} N(r)}{\log \frac{1}{1-r}} < \alpha_M + 2\varepsilon \quad \text{asymptotically,}$$

and $\sigma_{M,p+1}(f) \leq \alpha_M + 2\varepsilon$ for each $\varepsilon > 0$, i.e., $\sigma_{M,p+1}(f) \leq \alpha_M$ and $\gamma_{p+1} = \gamma_{M,p+1} \leq \alpha_M$.

On the other hand, let f_1, \dots, f_k be a solution base of Eq. (2.1). From the above discussion, we know that $i(f_j) \leq \delta \leq p + 1$. Then

$$\sigma_\delta(f_j) < \infty \quad (j = 1, \dots, k). \quad (4.1)$$

Hence, by Lemma 3.1, for some $\beta < \infty$,

$$m\left(r, \frac{f_j^{(n)}}{f_j}\right) = O(\log^+ T(r, f_j)) + O\left(\log \frac{1}{1-r}\right) = O\left(\exp^{[\delta-2]} \left\{ \left(\frac{1}{1-r} \right)^\beta \right\}\right) \\ (j = 1, \dots, k), \quad (4.2)$$

where $n \geq 1$ and $r \notin E$.

We now follow closely the method by H. Wittich using the standard order reduction procedure (see [10]); let us denote

$$v_1(z) = \frac{d}{dz} \left(\frac{f(z)}{f_1(z)} \right),$$

$A_k = 1$ and $v_1^{(-1)} = \frac{f}{f_1}$, i.e., $(v_1^{(-1)})' = v_1$. Hence

$$f^{(n)} = \sum_{m=0}^n \binom{n}{m} f_1^{(m)} v_1^{(n-1-m)} \quad (n = 0, \dots, k). \quad (4.3)$$

Substituting (4.3) into (2.1) and using the fact that f_1 solves (2.1), we obtain

$$v_1^{(k-1)} + A_{1,k-2}(z)v_1^{(k-2)} + \cdots + A_{1,0}(z)v_1 = 0, \quad (4.4)$$

where

$$A_{1,j} = A_{j+1} + \sum_{m=1}^{k-j-1} \binom{j+1+m}{m} A_{j+1+m} \frac{f_1^{(m)}}{f_1} \quad (4.5)$$

for $j = 0, \dots, k-2$. The meromorphic functions

$$v_{1,j}(z) = \frac{d}{dz} \left(\frac{f_{j+1}(z)}{f_1(z)} \right) \quad (j = 1, \dots, k-1) \quad (4.6)$$

form a solution base to (4.4). By (4.1) and Proposition 1.1(ii), the functions $v_{1,j}$ are of finite δ -order.

By (4.1), (4.5) and Proposition 1.1(ii), one can see that

$$m(r, A_{1,j}) = O\left(\exp^{[\delta-2]}\left\{\left(\frac{1}{1-r}\right)^\beta\right\}\right), \quad r \notin E, \quad j = 0, \dots, k-2, \quad (4.7)$$

implies

$$m(r, A_i) = O\left(\exp^{[\delta-2]}\left\{\left(\frac{1}{1-r}\right)^\beta\right\}\right), \quad r \notin E, \quad i = 0, \dots, k-1. \quad (4.8)$$

We may now proceed as above to further reduce the order of (4.4). In each reduction step, we obtain a solution base of meromorphic function of finite δ -order corresponding to (4.4), and the reasoning corresponding to (4.7) and (4.8) remains valid. Hence, we finally obtain an equation of type

$$v' + A(z)v = 0.$$

Since $\sigma_\delta(v) < \infty$,

$$m(r, A) = m\left(r, -\frac{v'}{v}\right) = O\left(\exp^{[\delta-2]}\left\{\left(\frac{1}{1-r}\right)^\beta\right\}\right), \quad r \notin E.$$

Observing the reasoning corresponding to (4.7) and (4.8) in each reduction step, we see that

$$m(r, A_j) = O\left(\exp^{[\delta-2]}\left\{\left(\frac{1}{1-r}\right)^\beta\right\}\right), \quad r \notin E \quad (4.9)$$

holds for $j = 0, \dots, k-1$. Since the coefficients A_j are analytic in Δ , we have $T(r, A_j) = O(\exp^{[\delta-2]}\{(\frac{1}{1-r})^\beta\})$ outside a possible exceptional set E . By Lemma 3.3, we have for any $c > 1$,

$$T(r, A_j) = O\left(\exp^{[\delta-2]}\left\{c\left(\frac{1}{1-r}\right)^\beta\right\}\right) = O\left(\exp^{[\delta-2]}\left\{\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right\}\right). \quad (4.10)$$

Hence, $\sigma_{\delta-1}(A_j) < \infty$, $j = 0, \dots, k-1$. Now $i(A_j) \leq \delta-1$ for all $j = 0, \dots, k-1$ and so $p \leq \delta-1$. Hence $\delta = p+1$.

Since $\delta = p+1 < \infty$, there exists a solution f of (2.1) such that $\sigma_{p+1}(f) = \gamma_{p+1}$. Since $\sigma_{p+1}(f) \leq \gamma_{p+1}$ for all solutions f of (2.1), we may replace β in the above reasoning with $\gamma_{p+1} + \varepsilon$, where $\varepsilon > 0$. By (4.10), we get

$$T(r, A_j) = O\left(\exp^{[p-2]}\left\{\left(\frac{1}{1-r}\right)^{\gamma_{p+1}+2\varepsilon}\right\}\right)$$

for all $j = 0, \dots, k-1$. Hence $\alpha \leq \gamma_{p+1}$.

Thus (ii) holds from the above discussion.

(iii) Set $p = 0$. Similar arguments to the first half of (ii), taking into account that $2r + 2rN_0(r) + \dots + 2rN_{k-1}(r) < (\frac{1}{1-r})^{m+1+\varepsilon}$ for all $1 > r > r_0(\varepsilon) > 0$, hence (iii) holds.

5. Proof of Theorem 2.2

By our assumptions, $\sigma_p(A_j) < \infty$ and $\sigma_{p-1}(A_j) = \infty$. Furthermore $\sigma_{p-1}(A_i) < \infty$ for $i = j+1, \dots, k-1$. Let f_1, \dots, f_{j+1} be linearly independent solutions of (2.1) such that $i(f_n) \leq p$.

Table 1

	k	$k-1$	\dots	$k-j$	\dots	j	$j-1$	\dots	1	0	solutions
v_0	1	$A_{0,k-1}$	\dots	$A_{0,k-j}$	\dots	$A_{0,j}$	$A_{0,j-1}$	\dots	$A_{0,1}$	$A_{0,0}$	$v_{0,1}, \dots, v_{0,j+1}$
v_1		1	\dots	$A_{1,k-j}$	\dots	$A_{1,j}$	$A_{1,j-1}$	\dots	$A_{1,1}$	$A_{1,0}$	$v_{1,1}, \dots, v_{1,j}$
\vdots				\vdots		\vdots	\vdots		\vdots	\vdots	\vdots
v_{j-1}				$A_{j-1,k-j}$	\dots	$A_{j-1,j}$	$A_{j-1,j-1}$	\dots	$A_{j-1,1}$	$A_{j-1,0}$	$v_{j-1,1}, v_{j-1,2}$
v_j				1	\dots	$A_{j,j}$	$A_{j,j-1}$	\dots	$A_{j,1}$	$A_{j,0}$	$v_{j,1}$

Hence $\sigma_p(f_n) < \infty$, $n = 1, \dots, j+1$. If $j = k-1$, then $\delta = \sup\{i(f): L(f) = 0\} \leq p$, contradicting Theorem 2.1(ii). Hence $j < k-1$.

We now apply the same order procedure as in the proof of Theorem 2.1. Let use the notation v_0 instead of f and $A_{0,0}, \dots, A_{0,k-1}$ instead of A_0, \dots, A_{k-1} . In the general reduction step, we obtain an equation of type

$$v_n^{(k-n)} + A_{n,k-n-1}(z)v_n^{(k-n-1)} + \dots + A_{n,0}(z)v_n = 0, \quad (5.1)$$

where

$$A_{n,j} = A_{n-1,j+1} + \sum_{m=1}^{k-j-n} \binom{j+1+m}{m} A_{n-1,j+1+m} \frac{v_{n-1,1}^{(m)}}{v_{n-1,1}} \quad (5.2)$$

and where the functions

$$v_{n,j}(z) = \frac{d}{dz} \left(\frac{v_{n-1,j+1}(z)}{v_{n-1,1}(z)} \right) \quad (j = 1, \dots, k-n)$$

determine a solution base of (5.1) in terms of the proceeding solution base.

We may express (2.1) and the j reduction steps by Table 1 as follows. The rows correspond to (5.1) for v_0, \dots, v_j , i.e., the first row corresponds to (2.1), and the column from k to 0 give the coefficients of these equations, while the last column lists the solutions of finite p -order.

By Proposition 1.2 and (5.2), we see that in the second row corresponding to the first reduction step, $m(r, A_{1,t}) = O(\exp^{[p-2]}(\frac{1}{1-r})^\beta)$, $r \notin E$, holds for $t = j, \dots, k-2$, while $i(A_{1,j-1}) = p$. Similarly, in each reduction step, Eq. (5.1) implies that $m(r, A_{n,t}) = O(\exp^{[p-2]}(\frac{1}{1-r})^\beta)$, $r \notin E$, holds for $t = j+1-n, \dots, k-(n+1)$, i.e., for all coefficients to the left from the boldface coefficient $A_{n,j-n}$, while $i(A_{n,j-n}) = p$ for $n = 1, \dots, j$. After j reduction steps, we have by (5.1)

$$A_{j,0} = -\frac{v_{j,1}^{(k-j)}}{v_{j,1}} - A_{j,k-j-1} \frac{v_{j,1}^{(k-j-1)}}{v_{j,1}} - \dots - A_{j,1} \frac{v'_{j,1}}{v_{j,1}}.$$

Hence, $m(r, A_{j,0}) = O(\exp^{[p-2]}(\frac{1}{1-r})^\beta)$, $r \notin E$. Since $A_{j,0}$ is analytic in Δ , $T(r, A_{j,0}) = m(r, A_{j,0})$ and by using Lemma 3.3 again, we obtain that $\sigma_{p-1}(A_{j,0}) < \infty$, contradicting $i(A_{j,0}) = p$. Therefore, Theorem 2.2 follows.

6. Proofs of Theorem 2.3 and 2.4

Proof of Theorem 2.3. Denote $\sigma_{M,p}(f) = \sigma$ and let $f \not\equiv 0$ be a solution of Eq. (2.1), then by (2.1), we have

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \dots + A_1 \frac{f'}{f}. \quad (6.1)$$

Hence,

$$|A_0| \leq \left| \frac{f^{(k)}}{f} \right| + \sum_{m=1}^{k-1} \left(|A_m| \left| \frac{f^{(m)}}{f} \right| \right). \quad (6.2)$$

Hence, by Lemma 3.5, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq O \left(\frac{T(1 - \beta(1 - r), f) \log^s(1 - r)}{(1 - r)^t} \right)^{2j}, \quad r \notin E, \quad (6.3)$$

where s, t are positive constants. By Lemma 3.4, there exists a set $H \subset (0, 1)$ with $\int_H \frac{dr}{1-r} = \infty$, for $|z| = r \in H$, given $\sigma - b > 2\varepsilon > 0$, where $b = \max\{\sigma_{M,p}(A_j) : j = 1, \dots, k-1\}$, then

$$M(r, A_0) \geq \exp^{[p]} \left\{ \left(\frac{1}{1-r} \right)^{\sigma-\varepsilon} \right\}. \quad (6.4)$$

Now, by (6.2)–(6.4), and

$$|A_j| \leq \exp^{[p]} \left\{ \left(\frac{1}{1-r} \right)^{b+\varepsilon} \right\},$$

we have

$$(1 - o(1)) \exp^{[p]} \left\{ \left(\frac{1}{1-r} \right)^{\sigma-\varepsilon} \right\} \leq O \left(\frac{T(1 - \beta(1 - r), f)}{(1 - r)^n} \right)^m, \quad r \in H - E,$$

where m, n are positive constants. So, by Lemma 3.3, $i(f) \geq p + 1$, and $\sigma_{p+1}(f) \geq \sigma$. By Theorem 2.1, we have $i(f) \leq p + 1$, and $\sigma_{p+1}(f) \leq \sigma$. Therefore, Theorem 2.3 follows. \square

Proof of Theorem 2.4. Denote $\sigma_p(A_0) = \sigma$ and set $f \neq 0$ be a solution of Eq. (2.1), then by (6.1), we have

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^k m \left(r, \frac{f^{(j)}}{f} \right) + O(1). \quad (6.5)$$

By Definition 1.3, we get that there exists $\{r'_n\}$ ($r'_n \rightarrow 1^-$) such that

$$\lim_{r'_n \rightarrow 1^-} \frac{\log^{[p]} T(r'_n, A_0)}{\log \frac{1}{1-r'_n}} = \sigma.$$

Set $\int_E \frac{dr}{1-r} = \log \delta < \infty$. Since

$$\int_{r'_n}^{1-(1-r'_n)/(\delta+1)} \frac{dr}{1-r} = \log(\delta + 1),$$

there exists

$$r_n \in \left[r'_n, 1 - \frac{1-r'_n}{\delta+1} \right] - E \subset [0, 1)$$

such that

$$\frac{\log^{[p]} T(r_n, A_0)}{\log \frac{1}{1-r_n}} \geq \frac{\log^{[p]} T(r'_n, A_0)}{\log \left(\frac{\delta+1}{1-r'_n} \right)} = \frac{\log^{[p]} T(r'_n, A_0)}{\log \frac{1}{1-r'_n} + \log(\delta + 1)}.$$

Hence

$$\lim_{r \rightarrow 1^-} \frac{\log^{[p]} T(r_n, A_0)}{\log \frac{1}{1-r_n}} \geq \lim_{r \rightarrow 1^-} \frac{\log^{[p]} T(r'_n, A_0)}{\log \frac{1}{1-r'_n} + \log(\delta + 1)} = \sigma.$$

Therefore

$$\lim_{r \rightarrow 1^-} \frac{\log^{[p]} T(r_n, A_0)}{\log \frac{1}{1-r_n}} = \sigma.$$

By the conditions of the theorem, for any given ε ($0 < 2\varepsilon < \sigma - b$), where $b = \max\{\sigma_p(A_j): j = 1, \dots, k-1\}$, we have

$$T(r_n, A_0) \geq \exp^{[p-1]} \left\{ \left(\frac{1}{1-r_n} \right)^{\sigma-\varepsilon} \right\}, \quad (6.6)$$

$$T(r_n, A_j) \leq \exp^{[p-1]} \left\{ \left(\frac{1}{1-r_n} \right)^{b+\varepsilon} \right\}. \quad (6.7)$$

By Lemma 3.1,

$$m\left(r_n, \frac{f^{(j)}}{f}\right) = O\left(\log\left(\frac{1}{1-r_n} T(r, f)\right)\right). \quad (6.8)$$

Hence we get from (6.5)–(6.8) that $i(f) \geq p+1$ and $\sigma = \sigma_p(A_0) \leq \sigma_{p+1}(f)$. By Theorem 2.1, we know that $i(f) \leq p+1$ and $\alpha_M \geq \sigma_{p+1}(f)$. Therefore, Theorem 2.4 follows. \square

7. Proof of Theorem 2.5

By Theorem 2.1, $\delta \leq 1 + p$. Set f_1, f_2 to be a fundamental system of solutions of Eq. (2.2), then the Wronskian $W = f_1 f'_2 - f'_1 f_2$ is analytic in Δ , and $\sigma_n(W) \leq \max\{\sigma_n(f_1), \sigma_n(f_2)\}$ by Proposition 1.1. Then

$$A_1(z) = -\frac{W'}{W} \quad \text{and} \quad A_0(z) = -\frac{f''}{f} - A_1 \frac{f'}{f} = -\frac{f''}{f'} \cdot \frac{f'}{f} - A_1 \frac{f'}{f}. \quad (7.1)$$

Furthermore, $\gamma_n = \max\{\sigma_n(f_1), \sigma_n(f_2)\}$ (because $f = af_1 + bf_2$, where f is a solution of (2.2), $a, b \in \mathbb{C}$). By (7.1), we obtain

$$m(r, A_1) = m\left(r, \frac{W'}{W}\right), \quad (7.2)$$

$$m(r, A_0) = m\left(r, \frac{f''}{f}\right) + 2m\left(r, \frac{f'}{f}\right) + m(r, A_1) + \log 2. \quad (7.3)$$

If $\delta = 0$, then $i(f) = 0$ for all solutions f , that is, every solution is a nonadmissible. But it is well known that this is possible only if $L(f) = f''$ and this case is excluded. Then $\delta = 1 + p$ is trivial if $p = 0$.

Let $p > 0$. By contradiction, let us assume that $0 < \delta < 1 + p$, then $\delta \leq p$ and $i(f) \leq p$. Hence, $\sigma_p(f_j) < \infty$ ($j = 1, 2$). So $\sigma_p(W) < \infty$. Then by Lemma 3.1 and (7.2), we get

$$T(r, A_1) = m(r, A_1) = m\left(r, \frac{W'}{W}\right) = O\left(\log\left(\frac{1}{1-r} T(r, W)\right)\right) \quad (r \notin E). \quad (7.4)$$

Since $\log^{[p]} T(r, W) = O(\log \frac{1}{1-r})$,

$$\log^{[p-1]} T(r, A_1) = O\left(\log\left(\frac{1}{1-r} T(r, W)\right)\right) = O\left(\log \frac{1}{1-r}\right) \quad (r \notin E). \quad (7.5)$$

By using Lemma 3.3 with $F(r) = \log^{[p-1]} T(r, A_1)$ and $G(r) = \log \frac{1}{1-r}$, we conclude that $\sigma_{p-1}(A_1) < \infty$ if $p > 1$, and A_1 is a nonadmissible if $p = 1$. But again from Lemma 3.1, by (7.3), we have

$$\begin{aligned} T(r, A_0) &= m(r, A_0) \\ &= O\left(\log\left(\frac{1}{1-r} T(r, f)\right)\right) + O\left(\log\left(\frac{1}{1-r} T(r, f')\right)\right) \\ &\quad + O\left(\log\left(\frac{1}{1-r} T(r, A_1)\right)\right) + O\left(\log\left(\frac{1}{1-r}\right)\right) \quad (r \notin E). \end{aligned} \quad (7.6)$$

Since $\sigma_p(f') = \sigma_p(f)$ by Proposition 1.1, we derive as above that $\sigma_{p-1}(A_0) < \infty$ if $p > 0$, or A_0 is nonadmissible if $p > 1$. Thus $p = \max\{i(A_0), i(A_1)\} \leq p-1$, this is a contradiction. Thus $\delta = 1 + p$.

(i) If $p = 0$, assume that $\gamma_1 < t$, where $t = \frac{D(A_0)-8}{4}$, then $D(A_0) > 8$, $\sigma(f') = \sigma(f) < d$ and $\sigma(W) < d$ for some $d \in (0, t)$. Thus from (7.3) and Lemma 3.6, we get

$$\begin{aligned} T(r, A_0) &= m(r, A_0) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| d\theta + \frac{1}{\pi} \int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{W'(re^{i\theta})}{W(re^{i\theta})} \right| d\theta + \log 2 \\ &\leq 4 \log^+ \left\{ \left(\frac{1}{1-r} \right)^{\sigma(f)+2+\varepsilon} \right\} + \log 2. \end{aligned}$$

By Lemma 3.3 again, we have $D(A_0) \leq 4d + 8 < 4t + 8 = D(A_0)$, a contradiction. Thus $\gamma_1 + 1 \geq \gamma_{M,1} \geq \gamma_1 \geq \frac{D(A_0)-8}{4}$, if $p = 0$.

(ii) If $p = 0$, assume that $\gamma_1 < s$, where $s = D(A_1) - 2 > 0$, then by similar arguments as in (i), from (7.2) and Lemma 3.6, we have a contradiction. Thus $\gamma_1 + 1 \geq \gamma_{M,1} \geq \gamma_1 \geq D(A_0) - 2$, if $p = 0$.

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